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Incompatibility of Gauge Invariance and Nonrelativistic Locality in Path Integral Formulation

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It is shown that a modification of the usual gauge transformations is essential to the path integral formulation of nonrelativistic quantum mechanics as a consequence of defining the locality condition as follows: The contribution from each path comes entirely from the points on the path. Arguments are based on the similarity between Wiener and Feynman functional (path) integrals.

KEY WORDS: Quantum mechanics; gauge invariance; path integral; stochastic integral.

Whether or not a quantum field theory is gauge-invariant and local is an important and well-known question. There seems to be no contradiction between these two requirements in field theory, nor quantum mechanics. It is therefore surprising to find that in the path integral formulation of quantum mechanics,[1] the situation may be different. More precisely, the usual gauge transformations of the wave function, scalar, and vector potentials do not leave invariant nonrelativistic quantum mechanics, where locality is defined by saying that *the contribution from each path comes entirely from the points on the path.* This definition appears very natural in the path integral formalism, however, as is explained below, it is more restrictive than the usual

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requirement that the potentials in the Schrödinger equation depend only on x and t.

The purpose of this note is to show the contradiction. Its origin is in the nondifferentiability of relevant paths in the Feynman (functional or path) integral.[2] A similar situation is known and well understood in the case of the Wiener integral in the theory of (real) continuous Markov processes.[2] Unfortunately, the mathematical theory of diffusion with complex "probability" has not been sufficiently developed and the results from the real case do not easily carry over to quantum mechanics. It is therefore difficult to foresee how the contradiction can be resolved. It is even more difficult to make any positive conclusion about the same question in the path integral formulation of quantum field theory, because in this case, the analogy with the real Wiener process is much less direct.[3]

For simplicity, let us consider the Schrödinger equation with scalar potential $V(x, t)$. The time-dependent Green's function of the equation can be written as an integral

$$K(x_1, t_1, x_2, t_2) = \int_{x_1}^{x_2} \exp \phi_{t_1}^{t_2}(x_t)\, d_F x_t, \qquad t_1 \leqslant t_2 \tag{1}$$

where the index F denotes the Feynman functional integration[1,3] over all continuous functions x_t of t with the initial and final values $x_{t_1} \equiv x_1$ and $x_{t_2} \equiv x_2$, respectively, the functional

$$\phi_{t_1}^{t_2}(x_t) = (i/h) \int_{t_1}^{t_2} V(x_t, t)\, dt \tag{2}$$

clearly being anti-Hermitian, additive, and homogeneous. In the absence of an interaction, $\phi_{t_1}^{t_2} \equiv 0$. Then, (1) becomes[1,3]

$$K(x_1, t_1, x_2, t_2) = [m/2\pi i\hbar(t_2 - t_1)]^{1/2} \times \exp[im(x_2 - x_1)^2/2\hbar(t_2 - t_1)] \tag{3}$$

In the path integral formalism, the usual gauge invariance is "derived" in the following way. The equation

$$d\lambda(x, t) - \operatorname{grad} \lambda(x, t)\, dx - [\partial\lambda(x, t)/\partial t]dt = 0 \tag{4}$$

is integrated along a path x_t between space–time points (x_1, t_1) and (x_2, t_2) and the result is added to (2).

[2] Among all continuous functions of time (paths) over which one integrates, there are so few differentiable ones that their contribution to the integral (1) is precisely zero.

One gets

$$\phi_{t_1}^{t_2}(x_t) = (i/h)\,\lambda(x_2\,,\,t_2) - (i/h)\,\lambda(x_1\,,\,t_1)$$
$$+ (i/h)\int_{t_1}^{t_2}\{V(x_t\,,\,t) - [\partial\lambda(x_t\,,\,t)/\partial t]\}\,dt$$
$$- (i/h)\int_{x_1}^{x_2}\operatorname{grad}\lambda(x_t\,,\,t)\,dx_t \tag{5}$$

The first two terms on the right can be brought outside the functional integral in (1) because they depend only on the initial and final points of the path x_t. They are the exponents of the factors which occur in the initial and final wave functions after the gauge transformation. Similarly, one identifies the last two terms in (5) to be the corresponding transformations of the scalar and vector potentials.

Our objection to the above derivation is based on two facts:

1. The identity (4) is correct only for differentiable paths x_t. Because we want to insist on the path locality as defined above, we cannot assume any approximate smoothness of the paths. Therefore (4) has to be replaced by another relation.

2. Analogy with real, continuous Markov processes. There, a similar (Wiener) integration is performed in the same space of all continuous functions x_t. Equation (4) is replaced there by a nontrivially different relation (Ito stochastic differential) which can be integrated along any continuous path (Ref. 2, Chapter 7; Ref. 4). Added to a functional similar to (2), it generates gaugelike transformations of the solution of the (real) diffusion equation and its potentials which differ essentially from the usual gauge transformations of quantum mechanics.[5] Because of the deep formal similarity between Wiener and Feynman integrals[3] and because of the coincidence of their functional spaces, it is highly unlikely that the necessary modification of (4) will be a trivial one. Unfortunately, very little is known about the Ito differential for the complex case.[5]

The present contradiction disappears in the classical limit. Indeed, the more classical our system is, the less important are all paths which differ from the classical trajectory.[1] In the limit, one is left with one differential path along which (4) and the complex Ito differential must coincide.

Finally, there remain three alternatives. The present contradiction is due to restricted applicability of either (i) nonrelativistic quantum mechanics or (ii) the path integral formulation, or (iii) it is a fundamental one.

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